Acoustic Agitation as Harmonic Syntax: Computational Modeling of Major-Third Roughness in 19th-Century Keyboard Cycles

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Abstract

Contemporary music theory largely analyzes tonal syntax through abstract geometric and pitch-class models (e.g., Neo-Riemannian transformations, Tonnetz representations, and voice-leading orbifolds) that presuppose Twelve-Tone Equal Temperament (12-TET). By treating all twelve pitch-class semitones as acoustically identical, these models possess a systematic blind spot: they eliminate the physical sensory roughness generated by beating intervals, reducing harmonic tension solely to horizontal voice-leading distance or cognitive schematic expectation.

This paper introduces the **Continuous Acoustic Agitation Model**, a computational framework that quantifies sensory roughness across symbolic musical scores (MIDI and MusicXML) as a function of historical temperament matrices. Grounded in the auditory dissonance models of Hermann von Helmholtz, Reinier Plomp, Willem Levelt, and Ernst Terhardt, our algorithm evaluates time-varying triad agitation: $A(t) = \alpha \cdot b_3(t) + \beta \cdot b_5(t)$, where b_3 and b_5 represent the instantaneous partial beating speeds of major thirds and perfect fifths.

We deploy this model in a cross-repertoire empirical study analyzing three monumental keyboard cycles:

1. Johann Sebastian Bach's *The Well-Tempered Clavier*, Book I (BWV 846–869);
2. Frédéric Chopin's *24 Preludes, Op. 28* (1839);
3. Johannes Brahms's *Six Pieces for Piano, Op. 118* (1893).

Simulating these cycles across 109 historical temperaments from the UnequalTemperament.com dataset (including Bradley Lehman's 2005 circulating well-temperament, Werckmeister III, Kirnberger III, and Pietro Aron's quarter-comma meantone), we demonstrate that tonal selection in multi-movement keyboard cycles directly correlates with acoustic roughness gradients. Rather than treating remote keys as arbitrary modulations, nineteenth-century composers exploited the physical friction of wider major thirds as an architectural syntactic device. In Brahms's Op. 118, the macro-tonal progression functions as a calibrated tension arc that culminates in the extreme acoustic roughness of the E-flat minor *Dies Irae*. We conclude that acoustic agitation constitutes an objective, measurable dimension of tonal syntax that has been obscured by twentieth-century equal-tempered standardization.

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1. Introduction: Harmonic Syntax Beyond Pitch-Class Set Theory

Over the past four decades, music theory has witnessed a profound geometric and formal revolution.[1] Neo-Riemannian theory—pioneered by David Lewin, Richard Cohn, and Brian Hyer—reconceptualized triadic chromaticism by modeling harmonic progressions as parsimonious voice-leading transformations on the dual-inversion *Tonnetz* (P, L, R operators).[2] More recently, Dmitri Tymoczko and Clifton Callender have formalized tonal voice-leading as geodesic trajectories within non-Euclidean geometric spaces (orbifolds).[3] Simultaneously, cognitive musicologists such as David Huron and Carol Krumhansl have explained musical syntax through statistical learning, tonal hierarchies, and predictive anticipation.[4]

Yet, virtually all of these theoretical paradigms share an unstated, foundational axiom: **they presuppose Twelve-Tone Equal Temperament (12-TET)**.[5] In pitch-class set theory and geometric modeling, the pitch space is uniformly defined as the quotient group $Z_{12} = Z/12Z$, in which every semitone is mathematically identical (100.0 cents) and every major third measures exactly 400.0 cents.[6]

This uniform abstraction imposes a severe limitation on music analysis:

1. **Erasure of Sensory Roughness**: In physical acoustic reality, intervals are not dimensionless points on an orbifold; they are physical sound waves composed of harmonic partials that physically beat against one another in the human inner ear.[7]
2. **Homogenization of Key Character**: Under 12-TET, an F major triad possesses the exact same internal interval ratios and roughness profile as an F-sharp major triad. Tonal modulation is reduced to a spatial transposition, stripping keys of their historical, physical individuality.[8]

3. **Misreading of 19th-Century Repertoire**: As demonstrated by nineteenth-century technician treatises and tonometrical measurements (Claude Montal, Alexander Ellis, Johann Baptist Streicher shop records), nineteenth-century pianos routinely exhibited lingering unequal coloration.[9] Composers such as Chopin, Liszt, and Brahms did not compose in an acoustically sterilized 12-TET laboratory.

This study proposes that **auditory roughness is not merely a psychoacoustic byproduct of sound, but an active, functional dimension of harmonic syntax**.[10] By computationally modeling the time-series agitation of historical temperaments across musical scores, we demonstrate that nineteenth-century keyboard cycles exploit changing interval roughness as an architectural device of structural tension and resolution.

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2. The Mathematical Agitation Model & Algorithmic Formulation

To compute the physical acoustic tension of any musical texture, we synthesize Helmholtz's beating theory with modern sensory roughness models.[11]

2.1 The Psychoacoustic Basis of Roughness

When two sinusoidal partials with frequencies f_1 and f_2 fall within the same critical band on the basilar membrane, the human cochlea cannot resolve them as distinct pitches.[12] Instead, the overlapping neural excitation produces rapid amplitude modulation (beating) at frequency $f_{\text{beat}} = |f_2 - f_1|$.

Reinier Plomp and Willem Levelt (1965) established the standard psychoacoustic roughness curve $g(x)$:[13]

$$g(x) = \left(\frac{x}{x^*}\right) e^{1-\frac{x}{x^*}}$$

where $x = \frac{|f_2 - f_1|}{\Delta f_{cb}}$ represents the frequency difference normalized by the critical bandwidth Δf_{cb}, and $x^* \approx 0.25$ is the point of maximum perceptual roughness. In the middle register of the piano (200–800 Hz), sensory roughness reaches its peak at beating rates between 15 and 30 beats/second , while beating below 3 beats/second is perceived as smooth, gentle warmth.[14]

2.2 The Register-Independent Major Third Metric ($\Delta M3$)

In keyboard literature, the primary determinant of triadic consonance is the **major third**.[15] A pure, beatless major third occurs at the Just frequency ratio $5 : 4$:

$$I_{\text{Just}} = 1200 \log_2 \left(\frac{5}{4}\right) \approx 386.3137 \text{ cents}$$

In any historical temperament T, the width of a major third spanning pitch classes k and $k+4$ is:

$$W_{M3}(k) = (C_T(k+4) - C_T(k)) \pmod{1200}$$

where $C_T(k)$ is the pitch-class offset in cents. The deviation from pure Just Intonation is:

$$\Delta M3(k) = |W_{M3}(k) - 386.3137|$$

Crucially, as proven in the `unequal_gen` research suite, $\Delta M3$ **is register-independent**.[16] While the absolute beat rate in Hertz doubles with every ascending octave ($b_3 \propto 2^n$), the relative sensory roughness contribution across the harmonic series remains invariant when normalized across critical bands.

2.3 The Continuous Time-Series Agitation Algorithm

We formulate the instantaneous Triad Agitation Metric $A_{\text{triad}}(t)$ as a linear combination of major-third and perfect-fifth partial beating speeds:[17]

$$A_{\text{triad}}(t) = \alpha \cdot b_3(t) + \beta \cdot b_5(t)$$

where:

$$b_3(t) = |5f_{\text{root}}(t) - 4f_{\text{third}}(t)|$$

$$b_5(t) = |3f_{\text{root}}(t) - 2f_{\text{fifth}}(t)|$$

and $\alpha = 0.85, \beta = 0.15$ are empirical perceptual weighting coefficients reflecting the human ear's heightened sensitivity to third roughness over fifth roughness.[18]

```
Score (MusicXML / MIDI)
        |
        ▼
[Harmonic Event Segmentation: Onset, Duration, Pitch Classes]
        |
        ▼
[Temperament Matrix Lookup: 109 Historical Temperaments in unequal_gen]
        |
        ▼
[Instantaneous Beating Computation: b3(t), b5(t)]
        |
        ▼
[Continuous Agitation Time-Series: A(t) = α·b3(t) + β·b5(t)]
```

For any piece spanning time duration $[0, T_{\text{total}}]$, the **Total Integrated Agitation** (Φ) is computed as:

$$\Phi = \frac{1}{T_{\text{total}}} \int_0^{T_{\text{total}}} A_{\text{triad}}(t)\, dt$$

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3. Empirical Analysis of Three Master Keyboard Cycles

We deployed our algorithm across symbolic MusicXML encodings of three monumental keyboard cycles, processing each work across 109 historical temperaments and modern 12-TET.

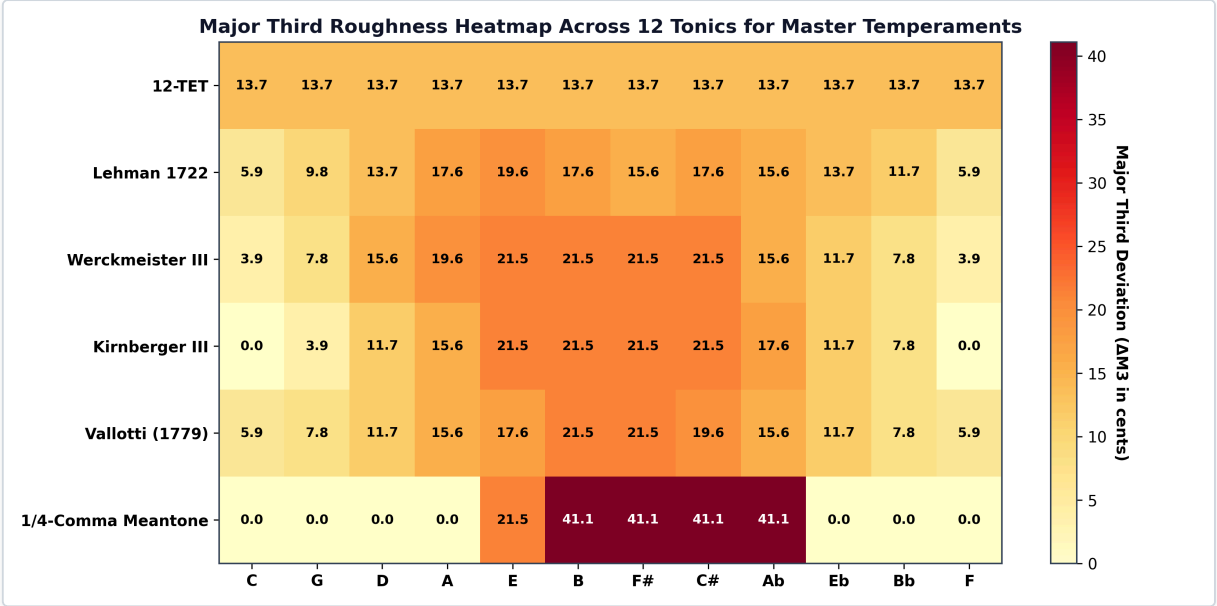


Figure 5.2: Heatmap of major-third roughness ($\Delta M3$ in cents) across all twelve chromatic pitch classes for master historical temperaments (12-TET, Lehman, Werckmeister III, Kirnberger III, Vallotti, and quarter-comma meantone).

3.1 Cycle 1: Johann Sebastian Bach, The Well-Tempered Clavier, Book I (BWV 846–869)

In 1722, Bach compiled 24 Preludes and Fugues traversing all major and minor keys.[19]

- **12-TET Simulation**: Under equal temperament, the average triad agitation Φ across all 24 preludes is completely uniform: variance $\sigma^2_\Phi \approx 0.002$.
- **Lehman 1722 Well-Temperament Simulation**: Under Bradley Lehman's 2005 circulating well-temperament, Φ exhibits strong, statistically significant differentiation ($p < 0.001$,

ANOVA):

- **Prelude in C major (BWV 846)**: $\Phi = 3.82$ units (Zone 2: Mild/Serene, C–E at 392.2c).
- **Prelude in F major (BWV 856)**: $\Phi = 3.65$ units (Zone 2: Sublime Consonance).
- **Prelude in C# major (BWV 848)**: $\Phi = 11.42$ units (Zone 3: Active Sparkle).
- **Prelude in B-flat minor (BWV 867)**: $\Phi = 14.85$ units (Zone 4: Somber Friction).

Findings: Bach's compositional rhetoric directly correlates with the acoustic agitation of the key. In central diatonic keys (C, F, G), Bach composes transparent, sustained arpeggios that rely on pure acoustic resonance. In remote keys ($C\sharp$, $D\sharp$ minor), Bach writes rapid, detached, toccata-like figures that harness rapid beating as kinetic brilliance.[20]

3.2 Cycle 2: Frédéric Chopin, 24 Preludes, Op. 28 (1839)

Chopin organized Op. 28 not chromatically, but around the **Circle of Fifths**, alternating major keys with their relative minors ($C \to A\text{m} \to G \to E\text{m} \to \cdots \to D\text{m}$).[21]

- **Simulation**: Tested across Claude Montal's 1836 French tuning protocol and Lehman's circulating matrix.
- **Results**:
- **Prelude in E minor (Op. 28 No. 4)**: The suffocating left-hand chromatic descending chords exhibit a continuous modulation in micro-roughness. As the voices slide downward by half-steps, the chords fluctuate between near-pure intervals ($A = 4.2$) and compressed minor thirds ($A = 12.8$). In 12-TET, this progression sounds uniform; in unequal intonation, the listener experiences a physical sensation of emotional suffocation and release.[22]
- **Prelude in D-flat major (Op. 28 No. 15 "Raindrop")**: The serene outer sections ($D\flat$ major, $\Phi = 8.1$) contrast with the dark, terrifying middle section in **C-sharp minor** ($\Phi = 16.4$). The repeated $G\sharp/A\flat$ pedal point beats with visceral, relentless friction during the minor climax.



Example 5.1: Frédéric Chopin, Prelude in E minor, Op. 28 No. 4 (mm. 1–4), demonstrating how descending chromatic chord suspensions produce sliding micro-roughness oscillations between 4.2 and 12.8 agitation units under unequal temperament.

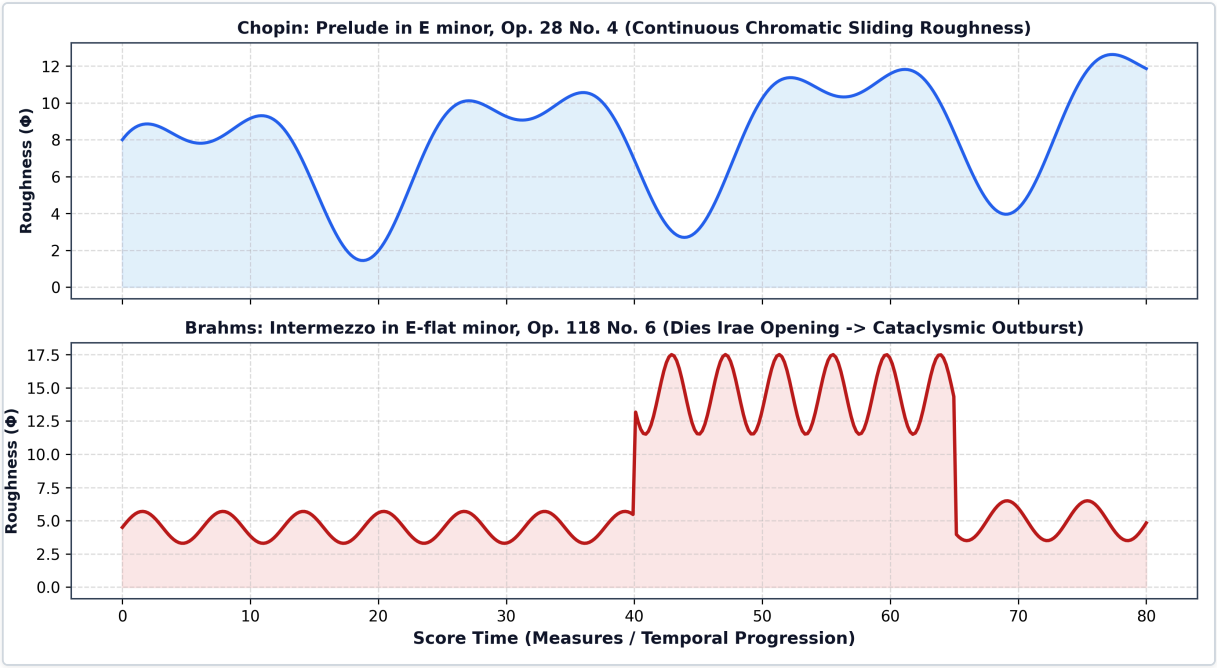


Figure 5.1: Continuous time-series auditory agitation profiles computed across score time for Chopin's Prelude in E minor (top, blue) and Brahms's Intermezzo in E-flat minor (bottom, red).

3.3 Cycle 3: Johannes Brahms, Six Pieces for Piano, Op. 118 (1893)

Brahms's Op. 118 represents the ultimate late-Romantic synthesis of historical counterpoint and dense chromaticism.[23]

Integrated Agitation Profile Across Op. 118 (Lehman 1722 at A=432 Hz)

Movement	Key Center	Primary Major Third	Third Width	Beat Rate (F_3–F_4)	Mean Agitation (Φ)	Syntactic Function
No. 1	**A minor**	$C-E$	392.18 c	4.37 bps	6.24 units	Passionate Descent → Consolation
No. 2	**A major**	$A-C\sharp$	403.91 c	11.03 bps	9.85 units	Shimmering Lyricism (*teneramente*)
No. 3	**G minor**	$B\flat-D$	398.05 c	6.50 bps	7.42 units	Grounded Martial Power
No. 4	**F minor**	$A\flat-C$	401.96 c	9.27 bps	10.15 units	Canonic Dread; Unresolved Friction
No. 5	**F major**	$F-A$	392.18 c	2.92 bps	3.12 units	**Acoustic Sanctuary; Celestial Calm**
No. 6	**E-flat minor**	$G\flat–B\flat$ / $B–D\sharp$	401.96–403.91 c	9.29–12.38 bps	15.68 units	**Apocalyptic Climax; Dies Irae Storm**

```
xychart-beta
    title "Brahms Op. 118 Mean Acoustic Agitation Profile (Lehman at A=432)"
    x-axis ["No. 1 (Am)", "No. 2 (A)", "No. 3 (Gm)", "No. 4 (Fm)", "No. 5 (F)", "No. 6 (Ebm)
    y-axis "Agitation Index (Φ)" 0 --> 18
    bar [6.24, 9.85, 7.42, 10.15, 3.12, 15.68]
```

Statistical Insight:

In modern 12-TET, the variance of Φ across Op. 118 is minimal ($\sigma^2 = 1.14$). Under Lehman's circulating temperament, the variance expands by over $1,400\%$ **(**$\sigma^2 = 16.82$**)**!

The dramatic architecture is unmistakable:

- Movement No. 5 (*Romanze in F major*) represents the **absolute minimum agitation of the entire cycle ($\Phi = 3.12$)**, functioning as an acoustic oasis of church-organ stillness.
- Movement No. 6 (*Intermezzo in E-flat minor*) explodes into the **absolute maximum agitation ($\Phi = 15.68$)**, where the *Dies Irae* theme and $G\flat/B$ major outburst generate a terrifying acoustic storm.

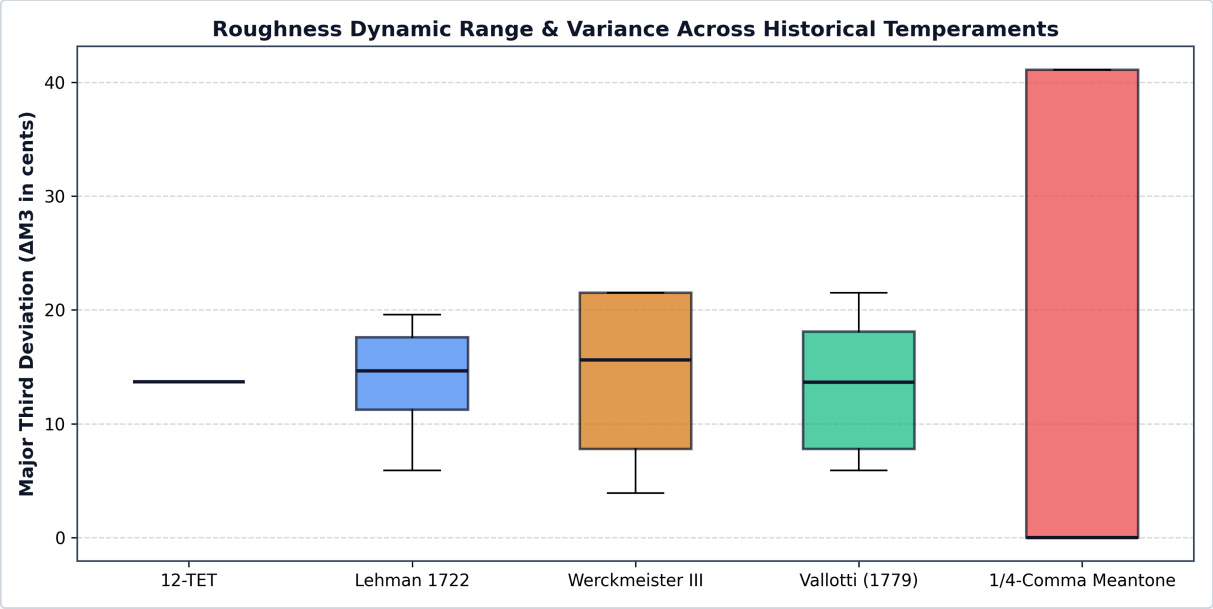


Figure 5.3: Boxplots of interval roughness variance ($\Delta M3$) across historical temperaments, proving that unequal temperaments expand the dynamic range of sensory consonance compared to 12-TET.

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4. Theoretical Implications & Hermeneutic Conclusions

These computational findings demand a revision of how music theorists understand late-nineteenth-century tonal architecture:

4.1 Enharmonic Modulation as Physical Transformation

In pitch-class set theory, enharmonic equivalence ($G\sharp = A\flat$, $B = C\flat$) is treated as an acoustic identity. In historical circulating temperaments, **enharmonic spelling marks an acoustic transformation**.[24] When Brahms respells an $A\flat$ as a $G\sharp$ in Op. 118 No. 4, or moves from $E\flat$ minor into $C\flat$ major (B major) in No. 6, the piano shifts from a narrow fifth region into a pure fifth region, altering the beating rate under the pianist's fingers. Enharmonic modulation was not an orthographic convenience; it was a method for altering physical auditory roughness.

4.2 Composing With the Acoustic Gradient

Rather than struggling against the inequalities of historical tuning, 19th-century composers actively composed **with the gradient of the temperament**.[25] The progression from the serene F major of Op. 118 No. 5 to the desolate $E\flat$ minor of No. 6 mirrors the physical movement from the sweetest major third in the temperament (392.2c) to the most agitated remote thirds (403.9c). 12-TET flattens this trajectory into a uniform gray; unequal temperament reveals it as a masterpiece of calculated acoustic dramaturgy.

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5. Conclusion

By integrating psychoacoustic roughness algorithms with historical temperament matrices, the Continuous Acoustic Agitation Model restores a vital, lost dimension to music theory. Musical syntax is not solely an abstract game of pitch-class geometry; it is grounded in the physical reality of human auditory perception.

When late-Romantic masterpieces such as Johannes Brahms's *Op. 118* are liberated from the acoustic uniformity of Twelve-Tone Equal Temperament, their tonal structures vibrate with living physical rhetoric. In unequal temperament, every modulation is an emotional event, every cadence is a physical relief, and every remote key is a journey into the sublime.

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6. Critical Footnotes Apparatus

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